

A CONCORDANCE EXTENSION THEOREM

JOEL L. JONES

ABSTRACT. Let $p: E \rightarrow B$ be a manifold approximate fibration between closed manifolds, where $\dim(E) \geq 4$, and let $M(p)$ be the mapping cylinder of p . In this paper it is shown that if $g: B \times I \rightarrow B \times I$ is any concordance on B , then there exists a concordance $G: M(p) \times I \rightarrow M(p) \times I$ such that $G|B \times I = g$ and $G|E \times \{0\} \times I = id_{E \times I}$. As an application, if N^n and M^{n+j} are closed manifolds where N is a locally flat submanifold of M and $n \geq 5$ and $j \geq 1$, then a concordance $g: N \times I \rightarrow N \times I$ extends to a concordance $G: M \times I \rightarrow M \times I$ on M such that $G|N \times I = g$. This uses the fact that under these hypotheses there exists a manifold approximate fibration $p: E \rightarrow N$, where E is a closed $(n+j-1)$ -manifold, such that the mapping cylinder $M(p)$ is homeomorphic to a closed neighborhood of N in M by a homeomorphism which is the identity on N .

1. INTRODUCTION

In this paper we extend concordances on the base on the mapping cylinder of an approximate fibration to the entire mapping cylinder. First we recall the relevant definitions.

If X is a topological space, then a *concordance on X* is a homeomorphism $g: X \times I \rightarrow X \times I$ such that $g_0 = id_X$. If $f: X \rightarrow Y$ is a continuous function of topological spaces, the *mapping cylinder of f* is the topological space

$$M(f) = \frac{(X \times [0, 1]) \amalg Y}{(x, 1) \sim f(x)}$$

with the quotient topology (where \amalg stands for the disjoint union).

All of our results depend on the notion of an approximate fibration, which was originally defined by Coram and Duvall [2] in terms of lifting properties. A map $p: E \rightarrow B$ has the *approximate homotopy lifting property for a space X* if given an open cover ε of B and maps $g: X \rightarrow E$ and $H: X \times I \rightarrow B$ such that $pg = H_0$, then there exists a map $G: X \times I \rightarrow E$ such that $G_0 = g$ and pG and H are ε -close. If the map G can always be chosen to be stationary with H , that is, for each $x_0 \in X$ so that $H(x_0, t)$ is a constant function of t , the function $G(x_0, t)$ is a constant function of t , then this is called the *regular approximate homotopy lifting property*. A proper map $p: E \rightarrow B$ of locally compact ANR's (i.e. absolute neighborhood retracts for the class of metric spaces) is an *approximate fibration* if it has the regular approximate homotopy lifting property for all spaces.

Received by the editors October 31, 1994.

1991 *Mathematics Subject Classification*. Primary 57N37; Secondary 55R65, 57N70.

Key words and phrases. Concordance, manifold approximate fibration, mapping cylinder.

At times this definition will be the most convenient one to use; however in most of our work we use a more recent definition of approximate fibration due to Hughes, Taylor, and Williams [8].

Let $p_i: E_i \rightarrow B$, $i = 0, 1$, be two spaces over B . A *controlled map* from p_0 to p_1 is a map $g: E_0 \times [0, 1) \rightarrow E_1 \times [0, 1)$ which is fibre preserving over $[0, 1)$ and such that the map $\bar{g}: E_0 \times [0, 1] \rightarrow B$, defined by $\bar{g}|_{E_0 \times [0, 1)} = p_1 \circ pr_{E_1} \circ g$ and $\bar{g}|_{E_0 \times \{1\}} = p_0$, is continuous. If this map g is a homeomorphism, then g is called a *controlled homeomorphism*. A map $p: E \rightarrow B$ is an *approximate fibration* if for every commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \cap & & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

there is a controlled map $\tilde{F}: X \times I \times [0, 1) \rightarrow E \times [0, 1)$ from F to p such that $\tilde{F}|_{X \times \{0\} \times [0, 1)} = f \times id_{[0, 1)}$.

That the two definitions for approximate fibrations differ will not present us with any problems. We will be able to move freely from one definition to the other because we will always have both E and B manifolds, and by Lemmas 12.10 and 12.11 of [8], the definitions are equivalent in this case. We will usually have the following situation.

A map $p: E \rightarrow B$ is a *manifold approximate fibration* if E and B are manifolds and p is a proper map which is an approximate fibration (in the sense of [8]). If E is a manifold with boundary, we also require that $\partial E = p^{-1}(\partial B)$.

Our main result is the following theorem. A proof is given in section 2

Concordance Extension Theorem. *Let $p: E \rightarrow B$ be a manifold approximate fibration between closed manifolds where $\dim(E) \geq 5$, and let $M(p)$ be the mapping cylinder of p . If $g: B \times I \rightarrow B \times I$ is any concordance on B , then there is a concordance $\tilde{g}: M(p) \times I \rightarrow M(p) \times I$ such that $\tilde{g}|_{B \times I} = g$ and $\tilde{g}|_{E \times \{0\} \times I} = id_{E \times I}$.*

From the work of Edwards [3], Pedersen [12], Quinn [13], and Hughes, Taylor, and Williams [9], we know that if N^n and M^{n+j} are closed manifolds, where $n \geq 5$ and $j \geq 1$, such that N is a locally flat submanifold of M , then there exists a manifold approximate fibration $p: E \rightarrow N$, where E is a closed $(n+j-1)$ -manifold, so that the mapping cylinder $M(p)$ is homeomorphic to a closed neighborhood of N in M by a homeomorphism which is the identity on N . This gives the following corollary as an application of the extension theorem.

Corollary (Concordance Extension Theorem for Topological Manifolds). *Let N^n be a locally flat submanifold of M^{n+j} where N and M are closed manifolds with $n \geq 5$ and $j \geq 1$. Suppose that $g: N \times I \rightarrow N \times I$ is a concordance on N . Then there is a concordance $\tilde{g}: M \times I \rightarrow M \times I$ on M such that $\tilde{g}|_{N \times I} = g$. \square*

The proof of the main extension theorem will be given in more generality than is necessary. There is the notion of spaces of concordances given in terms of simplicial sets (refer to May [11] for a detailed description of this type of object). For a manifold B , the *space of concordances on B* is defined to be the simplicial set $C(B)$ where a k -simplex of $C(B)$ is a homeomorphism $f: B \times I \times \Delta^k \rightarrow B \times I \times \Delta^k$

such that $f|B \times \{0\} \times \Delta^k = id$ and f commutes with projection on Δ^k , the standard k -simplex, and the boundary and degeneracy maps on $C(B)$ are induced by those on Δ^k .

With this in mind we say that given a continuous function of manifolds $p: E \rightarrow B$, the *space of concordances on $M(p)$* is defined to be the simplicial set $C(M(p))$, where a k -simplex of $C(M(p))$ is a homeomorphism $g: M(p) \times I \times \Delta^k \rightarrow M(p) \times I \times \Delta^k$ such that

- (1) g commutes with projection on Δ^k ,
- (2) $g|M(p) \times \{0\} \times \Delta^k = id$,
- (3) $g(E \times [0, 1) \times I \times \Delta^k) = E \times [0, 1) \times I \times \Delta^k$,
- (4) $g|E \times \{0\} \times I \times \Delta^k$ is a k -simplex of $C(E)$, and
- (5) $g|B \times I \times \Delta^k$ is a k -simplex of $C(B)$.

Here the boundary and degeneracy maps are also induced by those on Δ^k .

Perhaps we should use different notation in this definition, since here we keep track of much more relative information than we do for a concordance on just a manifold. It seems natural to want the restrictions to also be concordances, and so we will keep $C(M(p))$ for the notation. It should not generate any confusion.

It is easy to show that both of these simplicial sets satisfy the Kan condition since the simplicial structure is inherited from the Δ^k factor, and the standard simplicial complex is a Kan complex. We should also note that from characteristic (5) of $C(M(p))$ there is a function $\rho: C(M(p)) \rightarrow C(B)$ which is the restriction to the base of the mapping cylinder (this map is a simplicial map).

Using this terminology, the extension theorem states that if g is a k -simplex of $C(B)$, then there exists \tilde{g} , a k -simplex of $C(M(p))$, so that $\rho(\tilde{g}) = g$ and $\tilde{g}|E \times \{0\} \times I \times \Delta^k = id$.

The proof requires the following theorem about a *parametrized family of manifold approximate fibrations*, i.e. a manifold approximate fibration $p: M \times \Delta^k \rightarrow B \times \Delta^k$ which is fibre preserving over Δ^k , and such that $p_t: M \rightarrow B$ is a manifold approximate fibration for each t in Δ^k . It is a manifold with boundary version of Theorem 6.1 of [10]. The proof is given in section 3.

Controlled Isotopy Covering Theorem. *Let $p: M \times \Delta^k \rightarrow B \times \Delta^k$ be a k -parameter family of manifold approximate fibrations $M \rightarrow B$, $\dim(M) \geq 5$, with $p|\partial M \times \Delta^k \rightarrow \partial B \times \Delta^k$ being a k -parameter family of manifold approximate fibrations $\partial M \rightarrow \partial B$. Suppose that there is an isotopy $h: B \times \Delta^k \times I \rightarrow B \times \Delta^k \times I$ which is fibre preserving over Δ^k , and a continuous family of isotopies $g_s: \partial M \times \Delta^k \times I \rightarrow \partial M \times \Delta^k \times I$, $0 \leq s < 1$, such that*

- (1) $g_0 = id$,
- (2) g_s is fibre preserving over Δ^k for each s , and
- (3) $(p \times id_I)g_s$ converges uniformly to $h(p \times id_I)|\partial M \times \Delta^k \times I$ as $s \rightarrow 1^-$.

Then there is a continuous family of isotopies $H_s: M \times \Delta^k \times I \rightarrow M \times \Delta^k \times I$, $0 \leq s < 1$, such that

- (1) $H_0 = id$,
- (2) H_s is fibre preserving over Δ^k for each s ,
- (3) $H_s|\partial M \times \Delta^k \times I = g_s$ for each s , and
- (4) $(p \times id_I)H_s$ converges uniformly to $h(p \times id_I)$ as $s \rightarrow 1^-$.

The work presented in this paper is a revision of the author's doctoral dissertation written at Vanderbilt University.

2. CONCORDANCES ON MAPPING CYLINDERS

We begin this section with a lemma about mapping cylinders.

Lemma 2.1. *Suppose that for $i = 1, 2$, $f_i: PX_i \rightarrow Y_i$ is a continuous function and $h: Y_1 \rightarrow Y_2$ and $h: X_1 \rightarrow X_2$ are continuous functions such that $f_2 h = h f_1$. Then the function $H: M(f_1) \rightarrow M(f_2)$ defined by $H|X_1 \times [0, 1) = \tilde{h} \times id_{([0, 1])}$ and $H|Y_1 = h$ is a continuous function.*

Proof. For $i = 1, 2$ let $q_i: (X_i \times [0, 1]) \amalg Y_i \rightarrow M(f_i)$ be the quotient map. Suppose that $U \subset M(f_2)$ is an open set. Then

$$q_1^{-1}(H^{-1}(U)) = \tilde{h}^{-1}(q_2^{-1}(U) \cap (X_2 \times [0, 1])) \amalg h^{-1}(q_2^{-1}(U) \cap H_2)$$

is an open set. Hence $H^{-1}(U)$ is open and so H is a continuous function. \square

In this section we will show that given a k -simplex g of $C(B)$ there is a k -simplex \tilde{g} of $C(M(p))$ so that $\tilde{g}|B \times I \times \Delta^k = g$ and $\tilde{g}|E \times \{0\} \times I \times \Delta^k = id$. We accomplish this in two steps.

First note that if $p: E \rightarrow B$ is a manifold approximate fibration, then $(p \times id_{I \times \Delta^k}): E \times I \times \Delta^k \rightarrow B \times I \times \Delta^k$ is, too. If g is a k -simplex of $C(B)$, then $g(p \times id_{I \times \Delta^k}): E \times I \times \Delta^k \rightarrow B \times I \times \Delta^k$ is a manifold approximate fibration. We wish to obtain homeomorphisms $f: M(p \times id_{I \times \Delta^k}) \rightarrow M(g(p \times id_{I \times \Delta^k}))$ and $h: M(g(p \times id_{I \times \Delta^k})) \rightarrow M(p \times id_{I \times \Delta^k})$ such that, using the identification $M(p \times id_{I \times \Delta^k}) \equiv M(p) \times I \times \Delta^k$, hf is a k -simplex of $C(M(p))$.

The homeomorphism f is easy to obtain. Just define it as $f = id$ on $E \times I \times \Delta^k \times [0, 1)$ and $f = g$ on $B \times I \times \Delta^k$. It will be continuous by 2.1. The homeomorphism h will be constructed in several steps, finishing at Theorem 2.5. Our approach is suggested by the work of Hughes [6].

The first step is another lemma about mapping cylinders. It is only a restatement of Lemma 12.2 of [8] where the level preserving properties are disregarded. Let $p, q: X \rightarrow Y$ be maps between topological spaces. Let $v: X \times [0, 1) \rightarrow X \times [0, 1)$ be a map. Let $M(p)$ and $M(q)$ denote the mapping cylinders of p and q respectively. Define the function $P: X \times [0, 1] \rightarrow Y \times [0, 1]$ by extending $(p \times id_{[0, 1)})v: X \times [0, 1) \rightarrow Y \times [0, 1)$ via $q: X \times \{1\} \rightarrow Y \times \{1\}$. Define the function $V: M(q) \rightarrow M(p)$ by extending $v: X \times [0, 1) \rightarrow X \times [0, 1)$ via the identity on Y . Assume that Y is a regular space and that $p \times id_{[0, 1]}$ is a closed map.

Lemma 2.2. *P is continuous if and only if V is continuous.* \square

We should note that relative and k -parameter versions of this are trivially true.

For $i = 0, 1$, let $p_i: M \rightarrow B$ be approximate fibrations, $L \subset M$, and $p_0|L = p_1|L$. We say that p_0 and p_1 are *concordant rel L* if there is an approximate fibration $q: M \times [0, 1] \rightarrow B \times [0, 1]$ which restricts to p_i over $B \times \{i\}$ and such that $q|L \times [0, 1] = (p_0|L) \times id_{[0, 1]}$.

A k -parameter version of this would of course have an extra Δ^k factor in every set and require that p_0, p_1 , and q be fibre preserving over Δ^k .

Before we present an important “straightening” theorem, we need a lemma about approximate fibrations.

Lemma 2.3. *Suppose that $p: E \times [0, 1] \rightarrow B \times [0, 1]$ is an approximate fibration with $p_1 = p|E \times \{1\}: E \rightarrow B$ an approximate fibration. If $\bar{p}: E \times [0, \infty) \rightarrow B \times [0, \infty)$ is the map defined by $\bar{p}|E \times [0, 1] = p$ and $\bar{p}|E \times [1, \infty) = p_1 \times id_{[1, \infty)}$, then \bar{p} is an approximate fibration.*

Proof. Suppose we have a topological space X and maps f and F such that the following diagram commutes.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \times [0, \infty) \\ \cap & & \downarrow \bar{p} \\ X \times I & \xrightarrow{F} & B \times [0, \infty) \end{array}$$

Let $r: [0, \infty) \rightarrow [0, 1]$ be a retraction. Then we get another commutative diagram,

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{(id \times r)f} & E \times [0, 1] \\ \cap & & \downarrow p \\ X \times I & \xrightarrow{(id \times r)F} & B \times [0, 1] \end{array}$$

and so we get a controlled map $\widehat{F}: X \times I \times [0, 1] \rightarrow E \times [0, 1] \times [0, 1]$ from $(id \times r)F$ to p so that $\widehat{F}|X \times \{0\} \times [0, 1] = (id \times r)f \times id_{[0, 1]}$. Consider f, F and \widehat{F} in terms of component functions and define $\widetilde{F}: X \times I \times [0, 1] \rightarrow E \times [0, \infty) \times [0, 1]$ by

$$\widetilde{F}(x, t, s) = \begin{cases} \widehat{F}(x, t, s) & \text{if } (x, t) \in F^{-1}(B \times [0, 1]), \\ (\widehat{F}_1(x, t, s), F_2(x, t, s)) & \text{if } (x, t) \in F^{-1}(B \times [1, \infty)). \end{cases}$$

If $(x, 0) \in F^{-1}(B \times [0, 1])$, then $\widetilde{F}(x, 0, s) = \widehat{F}(x, 0, s) = (f(x), s)$ (since r is the identity on $[0, 1]$). If $(x, 0) \in F^{-1}(B \times [1, \infty))$, then

$$\widetilde{F}(x, 0, s) = (\widehat{F}_1(x, 0, s), F_2(x, 0, s)) = (f_1(x), f_2(x), s),$$

hence $\widetilde{F}|X \times \{0\} \times [0, 1] = f \times id_{[0, 1]}$.

To see that \widetilde{F} is a controlled map from F to \bar{p} , we only need to worry about $(x, t) \in F^{-1}(B \times [1, \infty))$, because the \widehat{F} portion is already a controlled map. For such an (x, t) , we have that $F_2(x, t) \in [1, \infty)$, so $(\bar{p} \circ pr_{E \times [0, \infty)} \circ \widetilde{F})(x, t, s) = \bar{p}(\widehat{F}(x, t, s), F_2(x, t)) = (p_1 \widehat{F}_1(x, t, s), F_2(x, t))$ which approaches $(F_1(x, t, s), F_2(x, t)) = F(x, t)$ as $s \rightarrow 1^-$. \square

Theorem 2.4. *Suppose that M and B are compact manifolds with boundary, $\dim(M) \geq 5$, $k > 0$, and for $i = 0, 1$, $p_i: M \times \Delta^k \rightarrow B \times \Delta^k$ is an approximate fibration which is fibre preserving over Δ^k ; then p_0 and p_1 are concordant rel $\partial M \times \Delta^k$ if and only if there is a homeomorphism of mapping cylinders $h: M(p_1) \rightarrow M(p_0)$ which is fibre preserving over Δ^k and which is the identity when restricted to $B \times \Delta^k$, $M \times \Delta^k \times \{0\}$, and $M(p_1)|\partial M \times \Delta^k$.*

Proof. If such a homeomorphism h exists, then we also get a homeomorphism by restricting; i.e. $h|: M \times \Delta^k \times [0, 1] \cong M \times \Delta^k \times [0, 1]$. By 2.2, $(p_0 \times id_{[0, 1]})h: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ extends continuously to a map $q: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ which restricts to p_i over $B \times \Delta^k \times \{i\}$, for $i = 0, 1$. This map is clearly fibre preserving over Δ^k and $q|\partial M \times \Delta^k \times [0, 1] = p_0|\partial M \times \Delta^k \times [0, 1] = p_1|\partial M \times \Delta^k \times [0, 1]$. To see that q is an approximate fibration, first note that $q|M \times \Delta^k \times [0, 1]$ is an approximate fibration. Any homotopy which lands in $B \times \Delta^k \times \{1\}$ can be moved into $B \times \Delta^k \times [0, 1]$ by a suitably small motion, and hence we can use the approximate homotopy lifting property on $B \times \Delta^k \times [0, 1]$ to

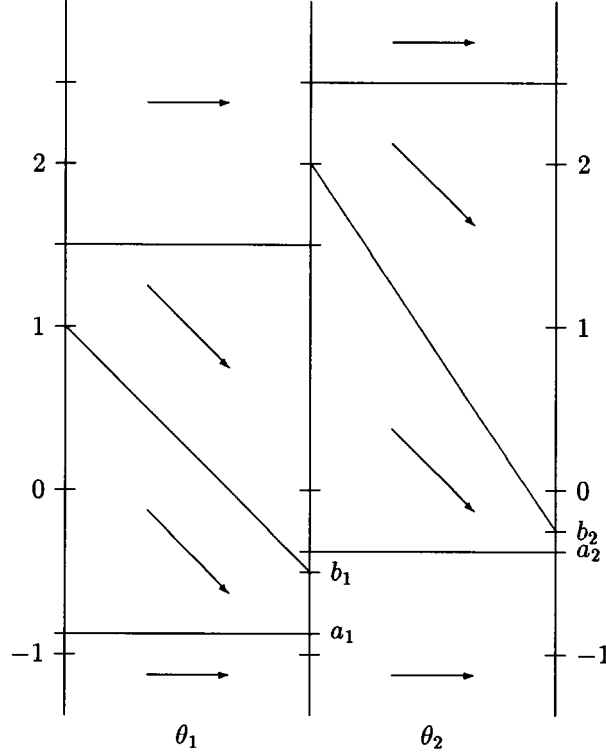


FIGURE 1

define an approximate homotopy lifting property on all of $B \times \Delta^k \times [0, 1]$. Hence p_0 and p_1 are concordant rel $\partial M \times \Delta^k$.

For the converse, let $q: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ be the concordance rel $\partial M \times \Delta^k$. Extend this to a map $Q: M \times \Delta^k \times \mathbf{R} \rightarrow B \times \Delta^k \times \mathbf{R}$ by defining $Q|M \times \Delta^k \times (-\infty, 0) = p_0 \times id_{(-\infty, 0)}: M \times \Delta^k \times (-\infty, 0) \rightarrow B \times \Delta^k \times (-\infty, 0)$ and $Q|M \times \Delta^k \times (1, \infty) = p_1 \times id_{(1, \infty)}: M \times \Delta^k \times (1, \infty) \rightarrow B \times \Delta^k \times (1, \infty)$. By 2.3, Q is an approximate fibration. Note that for each t in Δ^k , Q_t is an approximate fibration $M \times \mathbf{R} \rightarrow B \times \mathbf{R}$. We should also note that Q is level preserving in the \mathbf{R} factor on $M \times \Delta^k \times \mathbf{R}$ because $Q|\partial M \times \Delta^k \times \mathbf{R} = (p_0|\partial M \times \Delta^k) \times id_{\mathbf{R}} = (p_1|\partial M \times \Delta^k) \times id_{\mathbf{R}}$.

For each $i = 1, 2, \dots$, choose numbers a_i and b_i such that $\frac{-1}{i} < a_i < b_i < \frac{-1}{i+1}$. Let $\theta_i: \mathbf{R} \rightarrow \mathbf{R}$ be a homeomorphism satisfying

- (1) θ_i is supported on $[a_i, i + \frac{1}{2}]$,
- (2) θ_i maps $[a_i, i]$ linearly onto $[a_i, b_i]$, and
- (3) θ_i maps $[i, i + \frac{1}{2}]$ linearly onto $[b_i, i + \frac{1}{2}]$.

Figure 1 will help to explain the composition.

Let $\theta = \lim_{i \rightarrow \infty} \theta_i \circ \dots \circ \theta_1$. We note that θ exists and that $\theta: \mathbf{R} \rightarrow (-\infty, 0)$ is a homeomorphism which is supported on $[a_1, \infty)$.

The main effort in this proof will be spent in constructing a homeomorphism $H: M \times \Delta^k \times \mathbf{R} \rightarrow M \times \Delta^k \times (-\infty, 0)$ such that

- (1) $H|M \times \Delta^k \times (-\infty, -1] = id$,
- (2) H is fibre preserving over Δ^k ,

(3) $H|_{\partial M \times \Delta^k \times \mathbf{R}} = id \times \theta$, and

(4) $(p_0 \times id_{(-\infty, 0)})H: M \times \Delta^k \times \mathbf{R} \rightarrow B \times \Delta^k \times (-\infty, 0)$ extends continuously to a map $r: M \times \Delta^k \times (-\infty, \infty] \rightarrow B \times \Delta^k \times (-\infty, 0]$ by setting $r|M \times \Delta^k \times \{\infty\} = p_1: M \times \Delta^k \rightarrow B \times \Delta^k$.

Once we have this map H , we take a homeomorphism $\beta: [-1, 0) \rightarrow [0, 1)$, let $\alpha: [0, 1) \rightarrow [-1, \infty)$ be the homeomorphism $\alpha = (\theta^{-1}|_{[-1, 0)})\beta^{-1}$, and combine them with H in the following manner: $M \times \Delta^k \times [0, 1) \xrightarrow{id \times \alpha} M \times \Delta^k \times [-1, \infty) \xrightarrow{H} M \times \Delta^k \times [-1, 0) \xrightarrow{id \times \beta} M \times \Delta^k \times [0, 1) \xrightarrow{p_0 \times id} B \times \Delta^k \times [0, 1)$. Then this composition extends via p_1 and is fibre preserving over Δ^k . Now apply 2.2, and we obtain the desired mapping cylinder homeomorphism $h: M(p_1) \rightarrow M(p_0)$. Note that by (3) and the definition of α , $h|M(p_1)|_{\partial M \times \Delta^k} = id$.

Now to construct H . There is a simple isotopy $\tilde{\theta}_i$ from $id_{\mathbf{R}}$ to θ_i which is supported on $[a_i, i + \frac{1}{2}]$. Let $T_i = (id_{B \times \Delta^k}) \times \theta_i: B \times \Delta^k \times \mathbf{R} \rightarrow B \times \Delta^k \times \mathbf{R}$ and $T = \lim_{i \rightarrow \infty} T_i \circ \dots \circ T_1: B \times \Delta^k \times \mathbf{R} \rightarrow B \times \Delta^k \times (-\infty, 0)$ be the obvious homeomorphisms. For each i , there is the isotopy $\tilde{T}_i = (id_{B \times \Delta^k}) \times \tilde{\theta}_i$ from $id_{B \times \Delta^k \times \mathbf{R}}$ to T_i which is supported on $[a_i, i + \frac{1}{2}]$.

Let $\{\varepsilon_i\}_{i=1}^\infty$ be a sequence of positive numbers converging to zero fast enough so that $\varepsilon_i < \frac{-1}{i+1} - b_i$. For each i , let Θ^i be a one-parameter family of isotopies from $id_{\mathbf{R} \times I}$ to $\tilde{\theta}_i$ (e.g. for $0 \leq s \leq 1$, $\Theta_s^i = \tilde{\theta}_i|_{[0, s]}$). Then $f = (id_{\partial M \times \Delta^k}) \times \Theta^i: \partial M \times \Delta^k \times \mathbf{R} \times I \times [0, 1] \rightarrow \partial M \times \Delta^k \times \mathbf{R} \times I \times [0, 1]$ is a one-parameter family of isotopies on $\partial M \times \Delta^k \times \mathbf{R}$ such that for $0 \leq s \leq 1$,

- (1) $f_0^i = id$,
- (2) f_s^i is fibre preserving over Δ^k ,
- (3) f_s^i is supported on $[\frac{-1}{i}, i + 1] \subset \mathbf{R}$ (actually on $[a_i, i + \frac{1}{2}]$), and
- (4) $(Q \times id_I)f_s^i$ converges uniformly to $\tilde{T}_i((Q|\partial M \times \Delta^k \times \mathbf{R}) \times id_I)$.

By 3.8. (a supported version of the controlled isotopy covering theorem) we get a one-parameter family of isotopies $\mathcal{H}^i: M \times \Delta^k \times \mathbf{R} \times I \times [0, 1) \rightarrow M \times \Delta^k \times \mathbf{R} \times I \times [0, 1)$ satisfying similar properties, and for large enough s in $[0, 1)$, there is an isotopy \tilde{H}_i approximately covering the isotopy \tilde{T}_i so that the 1-level of \tilde{H}_i is a homeomorphism $H_i: M \times \Delta^k \times \mathbf{R} \rightarrow M \times \Delta^k \times \mathbf{R}$ such that

- (1) H_i is fibre preserving over Δ^k ,
- (2) H_i is supported on $[\frac{-1}{i}, i + 1] \subset \mathbf{R}$,
- (3) $H_i|\partial M \times \Delta^k \times \mathbf{R} = (id_{\partial M \times \Delta^k}) \times \theta_i$, and
- (4) QH_i is ε_i -close to T_iQ .

Property (3) is true because \mathcal{H}_s^i will agree with f_s^i on $\partial M \times \Delta^k \times \mathbf{R} \times I$.

There are several important remarks that need to be made here. Because of the sets on which these H_i 's are supported, $H_{i+1}H_i|M \times \Delta^k \times [i+1, \infty) = H_{i+1}|M \times \Delta^k \times [i+1, \infty)$. Since we also chose the ε_i 's sufficiently small, $H_i(M \times \Delta^k \times (-\infty, i]) \subset M \times \Delta^k \times (-\infty, \frac{-1}{i+1}]$, and hence $H_{i+1}H_i|M \times \Delta^k \times (-\infty, i] = H_i|M \times \Delta^k \times (-\infty, i]$.

We claim that $H = \lim_{i \rightarrow \infty} H_i \circ \dots \circ H_1$ is the desired homeomorphism. First it must be seen that this limit exists. For $j \geq i$, $H_j \circ \dots \circ H_1|M \times \Delta^k \times (-\infty, i] = H_i \circ \dots \circ H_1|M \times \Delta^k \times (-\infty, i]$. Hence $\lim_{j \rightarrow \infty} (H_j \circ \dots \circ H_1)|M \times \Delta^k \times (-\infty, i] = H_i \circ \dots \circ H_1|M \times \Delta^k \times (-\infty, i]$, and so $\lim_{i \rightarrow \infty} H_i \circ \dots \circ H_1$ exists.

Next there several properties that must be verified. We have that $\text{im}(H) = M \times \Delta^k \times (-\infty, 0)$, because if $y = (y_1, y_2, y_3) \in M \times \Delta^k \times (-\infty, 0)$, then for some i , $y_3 \leq \frac{-1}{i+1}$; hence there is an $x \in M \times \Delta^k \times (-\infty, i]$ so that $H_i(x) = y$, (since H_i

is a homeomorphism for each i). Since $H_j \circ \cdots \circ H_1 = H_i \circ \cdots \circ H_1$ for $j \geq i$, we have that $y \in \text{im}(H_j \circ \cdots \circ H_1)$. Hence $y \in \text{im}(H)$.

These same properties are used to show that H is injective, in particular noting that $H|M \times \Delta^k \times (-\infty, i] = H_i \circ \cdots \circ H_1|M \times \Delta^k \times (-\infty, i]$, and so $H_i \circ \cdots \circ H_1$ is injective.

To see that H is a homeomorphism we note that $H^{-1}(M \times \Delta^k \times (-\infty, \frac{-1}{i}]) \subset H_i^{-1}(M \times \Delta^k \times (-\infty, \frac{-1}{i}]) \subset M \times \Delta^k \times (-\infty, i]$, and so $H|H^{-1}(M \times \Delta^k \times (-\infty, \frac{-1}{i}]) = H_i \circ \cdots \circ H_1|H^{-1}(M \times \Delta^k \times (-\infty, \frac{-1}{i}])$ which is a homeomorphism. Now since $\text{im}(H) = M \times \Delta^k \times (-\infty, 0)$ and H is injective, we have that H is a homeomorphism.

From the construction of H , it is easy to see that $H|M \times \Delta^k \times (-\infty, i] = \text{id}$, H is fibre preserving over Δ^k , and $H|\partial M \times \Delta^k \times \mathbf{R} = (\text{id}_{\partial M \times \Delta^k}) \times \theta$.

We have only to verify the extension property for H . For $i > 1$,

$$\begin{aligned} H_i H_{i-1}|M \times \Delta^k \times [i-1, i] &= H_i \circ \cdots \circ H_1|M \times \Delta^k \times [i-1, i] \\ &= H|M \times \Delta^k \times [i-1, i] \subset M \times \Delta^k \times (-\infty, 0]. \end{aligned}$$

Hence

$$\begin{aligned} (p_0 \times \text{id}_{[i-1, i]})H|M \times \Delta^k \times [i-1, i] &= QH|M \times \Delta^k \times [i-1, i] \\ &= QH_i H_{i-1}|M \times \Delta^k \times [i-1, i] \end{aligned}$$

which is $(\varepsilon_i + \varepsilon_{i-1})$ -close to

$$\begin{aligned} T_i T_{i-1} Q|M \times \Delta^k \times [i-1, i] &= (T_i \circ \cdots \circ T_1)Q|M \times \Delta^k \times [i-1, i] \\ &= (T_i \circ \cdots \circ T_1)(p_1 \times \text{id}_{[i-1, i]})|M \times \Delta^k \times [i-1, i]. \end{aligned}$$

Hence if we let $i \rightarrow \infty$, $(\varepsilon_i + \varepsilon_{i-1}) \rightarrow 0$, and so $(p_0 \times \text{id})H$ extends via p_1 as desired. \square

Now back to our original situation, where E and B are manifolds (in particular without boundary), $\dim(E) \geq 4$, $p: E \rightarrow B$ is a manifold approximate fibration and g is a k -simplex of $C(B)$. We see that $p \times \text{id}_{I \times \Delta^k}$ and $g(p \times \text{id}_{I \times \Delta^k})$ are manifold approximate fibrations $E \times I \times \Delta^k \rightarrow B \times I \times \Delta^k$, and that $g(p \times \text{id}_{I \times \Delta^k})|E \times \{0\} \times \Delta^k = p \times \text{id}_{I \times \Delta^k}$.

Theorem 2.5. *There is a homeomorphism $h: M(g(p \times \text{id}_{I \times \Delta^k})) \rightarrow M(p \times \text{id}_{I \times \Delta^k})$ such that $h|E \times I \times \Delta^k \times \{0\} = \text{id}$, $h|B \times I \times \Delta^k = \text{id}$, $h|M(p \times \text{id}_{I \times \Delta^k}) = \text{id}$, and h commutes with projection on Δ^k .*

Proof. We first see that the manifold approximate fibrations $p \times \text{id}_{I \times \Delta^k}$ and $g(p \times \text{id}_{I \times \Delta^k})$ are concordant rel $E \times \{0\} \times \Delta^k$ by a rotation as illustrated in Figure 2.

From here the proof closely follows the proof of 2.4. The first difference is in the homeomorphism H we wish to construct. It should satisfy a different property (3), namely $H|E \times \{0\} \times \Delta^k \times \mathbf{R} = (\text{id}_{E \times \Delta^k}) \times \theta$.

Define θ_i , $\tilde{\theta}_i$, Θ^i , T_i , \tilde{T}_i , etc., as before. The next major difference arises in obtaining the approximate covering isotopies. In order to use 3.8, we should already have some continuous family of isotopies defined on $\partial(E \times I) \times \Delta^k \times \mathbf{R}$. Since there are two boundary components here, we must define this family in two parts. On $E \times \{0\} \times \Delta^k \times \mathbf{R}$, the family will be $f_{0,s}^i = (\text{id}_{E \times \Delta^k}) \times \Theta_s^i$ for $0 \leq s < 1$, and on $E \times \{1\} \times \Delta^k \times \mathbf{R}$, use the $\partial M = \emptyset$ version of 3.8 to obtain some continuous family $f_{1,s}^i$ of isotopies (see also 6.1 of [10]). Then $f_s^i = f_{0,s}^i \cup f_{1,s}^i$ is an appropriate family

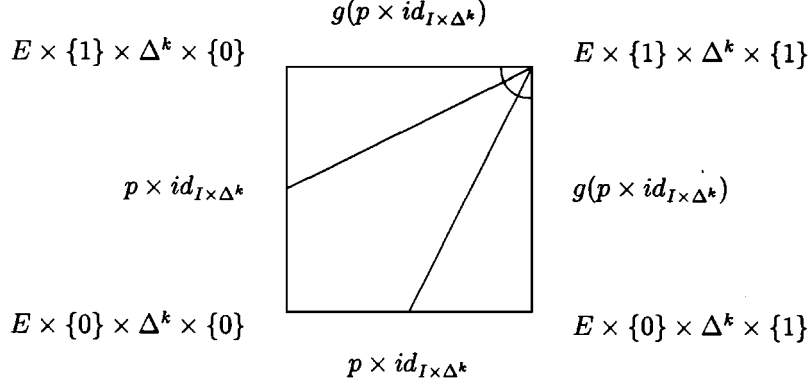


FIGURE 2

of isotopies on $\partial(E \times I) \times \Delta^k \times \mathbf{R}$ to which we can apply 3.8. Do this for each $i = 1, 2, \dots$, and obtain an isotopy \tilde{H}_i whose 1-level, H_i , has the same properties as those obtained in 3.8, except for a weakened (3), $H_i|E \times \{0\} \times \Delta^k \times \mathbf{R} = (id_{E \times \Delta^k}) \times \theta_i$. The resulting function $H = \lim_{i \rightarrow \infty} H_i \circ \dots \circ H_1$ is the desired homeomorphism. \square

This theorem gives us the long awaited homeomorphism $h: M(g(p \times id_{I \times \Delta^k})) \rightarrow M(p \times id_{I \times \Delta^k})$, and so combined with the homeomorphism $f: M(p \times id_{I \times \Delta^k}) \rightarrow M(g(p \times id_{I \times \Delta^k}))$, we have the following theorem.

Theorem 2.6. *Suppose that $p: E \rightarrow B$ is a manifold approximate fibration between closed manifolds, where $\dim(E) \geq 4$; then for every k -simplex g of $C(B)$ there is a k -simplex \tilde{g} of $C(M(p))$ so that $\tilde{g}|B \times I \times \Delta^k = g$ and $\tilde{g}|E \times \{0\} \times I \times \Delta^k = id$. \square*

3. THE CONTROLLED ISOTOPY COVERING THEOREM FOR MANIFOLDS WITH BOUNDARY

In this section we prove the following isotopy covering theorem. It is a generalization of one presented in section 6 of [10]. Many of the approximate fibration results, most notably 3.1, 3.2, and 3.8, that follow can be traced back to the non-parameterized versions of Chapman [1].

Theorem 3.1 (The controlled isotopy theorem for manifolds with boundary). *Let $p: M \times \Delta^k \rightarrow B \times \Delta^k$ be a k -parameter family of manifold approximate fibrations $M \rightarrow B$, $\dim(M) \geq 5$, with $p|\partial M \times \Delta^k \rightarrow \partial B \times \Delta^k$ being a k -parameter family of manifold approximate fibrations $\partial M \rightarrow \partial B$. Suppose that there is an isotopy $h: B \times \Delta^k \times I \rightarrow B \times \Delta^k \times I$ which is fibre preserving over Δ^k , and a continuous family of isotopies $g_s: \partial M \times \Delta^k \times I \rightarrow \partial M \times \Delta^k \times I$, $0 \leq s < 1$, such that*

- (1) $g = id$,
- (2) g is fibre preserving over Δ^k for each s , and
- (3) $(p \times id_I)g_s$ converges uniformly to $h(p \times id_I)|\partial M \times \Delta^k \times I$ as $s \rightarrow 1^-$.

Then there is a continuous family of isotopies $H_s: M \times \Delta^k \times I \rightarrow M \times \Delta^k \times I$, $0 \leq s < 1$, such that

- (1) $H_0 = id$,
- (2) H_s is fibre preserving over Δ^k for each s ,

- (3) $H_s|_{\partial M \times \Delta^k \times I} = g_s$ for each s , and
- (4) $(p \times id_I)H_s$ converges uniformly to $h(p \times id_I)$ as $s \rightarrow 1^-$.

The techniques we use to prove this theorem are mainly an adaptation of the work of Hughes, Taylor, and Williams [8] to the manifold with boundary case. To begin we assume the following α -straightening theorem for manifolds with boundary. The proof follows from the proof of the $\partial M = \emptyset$ case of [5].

Theorem 3.2. *Let B be a manifold, U an open subset of B , $\dim(M) \geq 5$, and $k \geq 0$. For every open cover α of B there exists an open cover β of B such that if M is any manifold with boundary ∂M , $p: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ is a proper fibre preserving map over $\Delta^k \times [0, 1]$ such that*

- (1) *for each t in $\Delta^k \times [0, 1]$, $p_t: M \rightarrow B$ and $p_t|_{\partial M}: \partial M \rightarrow B$ are approximate fibrations,*
 - (2) *$\partial M \times \Delta^k \times [0, 1] \subset p^{-1}(U \times \Delta^k \times [0, 1])$, and*
 - (3) *$G: p^{-1}(U \times \Delta^k \times [0, 1]) \rightarrow M \times \Delta^k \times [0, 1]$ is a fibre preserving open embedding so that $G|_{p^{-1}(U \times \Delta^k \times \{0\})} = id$ and pG is β -close to $p|_{p^{-1}(U \times \Delta^k \times \{0\})} \times id_{[0, 1]}$.*
- Then there exists a fibre preserving homeomorphism $H: M \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1]$, such that*
- (1) *$H|M \times \Delta^k \times \{0\} = id$,*
 - (2) *pH is α -close to $(p|M \times \Delta^k \times \{0\}) \times id_{[0, 1]}$,*
 - (3) *$H = G$ in a neighborhood of $\partial M \times \Delta^k \times [0, 1]$.* □

We need to introduce several theorems at this point which will be used to prove an important lemma.

Theorem 3.3 (Hu [4, p. 112]). *If Y is an ANR and α is an open cover of Y , then there exists an open cover β of Y , which is an open refinement of α , such that for any two β -near maps $f, g: X \rightarrow Y$ defined on a metrizable space X and any β -homotopy $j_t: A \rightarrow Y$, $0 \leq t \leq 1$, defined on a closed subspace A of X with $j_0 = f|_A$ and $j_1 = g|_A$, there exists an α -homotopy $h_t: X \rightarrow Y$, $0 \leq t \leq 1$, such that $h_0 = f$, and $h_1 = g$, and $h_t|_A = j_t$ for every t , $0 \leq t \leq 1$.* □

Theorem 3.4 (Hughes-Taylor-Williams [8, p. 53]). *Let B be a manifold, let $m \geq 5$, $k \geq 0$, and let α be an open cover of B . There is an open cover β of B such that if M is an m -manifold with boundary ∂M and $f: M \times \Delta^k \rightarrow B \times \Delta^k$ is a proper map which is fibre preserving over Δ^k such that $f_t: M \rightarrow B$ is a β -fibration for each t in $\partial \Delta^k$, and an approximate fibration over an open subset of B containing $f_t(\partial M)$, then there is a fibre preserving (over Δ^k) approximate fibration $F: M \times \Delta^k \rightarrow B \times \Delta^k$ such that F is α -close to f_t for each t in Δ^k and $F = f$ on $(M \times \partial \Delta^k) \cup (\partial M \times \Delta^k)$.* □

Lemma 3.5. *Let M and B be manifolds with $\dim(M) \geq 5$. For any $\varepsilon > 0$ there exists an open cover δ of B such that if $f, g: M \times \Delta^k \rightarrow B \times \Delta^k$ are δ -close k -parameter families of approximate fibrations $M \rightarrow B$, and if there exists a δ -homotopy $j: \partial M \times \Delta^k \rightarrow \partial B \times \Delta^k$ such that $j_0 = f|_{\partial M \times \Delta^k}$ and $j_1 = g|_{\partial M \times \Delta^k}$ and j_s is a k -parameter family of approximate fibrations $\partial M \rightarrow \partial B$ for $0 \leq s \leq 1$, then there exists an ε -homotopy $h: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ such that for $0 \leq s \leq 1$, h_s is a k -parameter family of approximate fibrations $M \rightarrow B$, $h_0 = f$, $h_1 = g$, and $h_s|_{\partial M \times \Delta^k} = j_s$.*

Proof. For $k = 0$, let β be an open cover as guaranteed by 3.4 corresponding to an open cover of B by $\frac{\varepsilon}{2}$ -balls, and let δ be an open cover of B as guaranteed by 3.3

which is a refinement of β . By 3.3 we get a β -homotopy $h: M \times \Delta^1 \rightarrow B \times \Delta^1$ which is an approximate fibration on $M \times \partial\Delta^1$, a β -fibration on $M \times \Delta^1$, and we can extend j to a collar neighborhood of ∂M , so that h agrees with j on some closed neighborhood of $\partial M \times \Delta^1$, and now 3.4 applies and we get a fibre preserving approximate fibration $H: M \times \Delta^1 \rightarrow B \times \Delta^1$.

Assume now that $k \geq 1$ and that the theorem is true for $k-1$. In particular, this means that we have approximate fibrations on $M \times \partial\Delta^k$, along with the approximate fibration we are given on $\partial M \times \Delta^k$. Now apply 3.3 and 3.4 as shown above to obtain the result. \square

Theorem 3.6. *Let M and B be manifolds with boundary, $\dim(M) \geq 5$, and let $p: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ be a proper fibre preserving map over $\Delta^k \times [0, 1]$ such that $p_t: M \rightarrow B$ and $p_t|_{\partial M}: \partial M \rightarrow \partial B$ are approximate fibrations for each t in $\Delta^k \times [0, 1]$. Let U be a neighborhood of ∂B and let V be a neighborhood of ∂M such that $V \times \Delta^k \times [0, 1] \subset p^{-1}(U \times \Delta^k \times [0, 1])$. Suppose that there is an open embedding $G: V \times \Delta^k \times [0, \infty) \rightarrow M \times \Delta^k \times [0, \infty)$ such that*

- (1) $G|_{V \times \Delta^k \times \{0\}} = id$,
- (2) G is fibre preserving over $\Delta^k \times [0, \infty)$, and
- (3) $(p_1|_{V \times \Delta^k \times [0, \infty)})G: V \times \Delta^k \times [0, \infty) \rightarrow B \times \Delta^k \times [0, \infty)$ extends to a map $V \times \Delta^k \times [0, \infty) \rightarrow B \times \Delta^k \times [0, \infty)$ via $p_0|$, where $p_i = p|_{M \times \Delta^k \times \{i\}}$.

Then there is a homeomorphism $H: M \times \Delta^k \times [0, \infty) \rightarrow M \times \Delta^k \times [0, \infty)$ such that

- (1) $H|_{M \times \Delta^k \times \{0\}} = id$.
- (2) H is fibre preserving over $\Delta^k \times [0, \infty)$,
- (3) $H|_{\partial M \times \Delta^k \times [0, \infty)} = G|_{\partial M \times \Delta^k \times [0, \infty)}$, and
- (4) $(p_1 \times id_{[0, \infty)})H: M \times \Delta^k \times [0, \infty) \rightarrow B \times \Delta^k \times [0, \infty)$ extends to a map $M \times \Delta^k \times [0, \infty) \rightarrow B \times \Delta^k \times [0, \infty)$ via p_0 .

Proof. The technique used in this proof comes from Hughes [6]. Choose a sequence $\{\varepsilon_i\}_{i=0}^\infty$ of positive numbers converging to zero. For each i , let δ_i be an open cover of B as found by the previous lemma. Furthermore assume that each member of δ_i has diameter less than ε_i . For each $i = 0, 1, 2, \dots$ we will construct a homeomorphism $h^i: M \times \Delta^k \times [i, i+1] \rightarrow M \times \Delta^k \times [i, i+1]$ such that

- (1) h^i is fibre preserving over $\Delta^k \times [i, i+1]$,
- (2) $h^0|_{M \times \Delta^k \times \{0\}} = id$, and if $i > 0$, $h^i|_{M \times \Delta^k \times \{i\}} = h^{i-1}|_{M \times \Delta^k \times \{i\}}$,
- (3) $(p_1 \times id_{[i, i+1]})h^i$ is $(\varepsilon_i + \delta_{i+1})$ -close to $p_0 \times id_{[i, i+1]}$ if $i > 0$,
- (4) $p_1 h^i|_{M \times \Delta^k \times \{i+1\}}$ is δ_{i+1} -close to p_0 , and
- (5) $h^i = G$ in a neighborhood of $\partial M \times \Delta^k \times [i, i+1]$.

We then piece together these homeomorphisms in this obvious manner to define the function H .

Define $a_i: M \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [i, i+1]$ by $a_i(m, s, t) = (m, s, t+i)$; let $G_i = G|_{V \times \Delta^k \times \{i\}}$ and let $g^i = a_i^{-1}(G_i \times id_{[i, i+1]})(G|_{V \times \Delta^k \times [i, i+1]})a_i$. Then $g^i: V \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1]$ is an open embedding which is fibre preserving over $\Delta^k \times [0, 1]$, has $g^i|_{V \times \Delta^k \times \{0\}} = id$, and $pg^i|_{\partial M \times \Delta^k \times [0, 1]} \rightarrow \partial B \times \Delta^k \times [0, 1]$ is an ε_i -homotopy from $p_0|$ to $p_1 g^i|$ through k -parameter families of approximate fibrations. To achieve this we may need to reparametrize G in the $[0, \infty)$ factor.

The homeomorphism $h^0: M \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1]$ follows directly from 3.2. Assume that $i > 0$ and h^{i-1} is already defined. Since $p_1 h^{i-1}|_{M \times \Delta^k \times \{i\}}$

is δ_i -close to p_o and since $pg^{i-1}|: \partial M \times \Delta^k \times [0, 1] \rightarrow \partial B \times \Delta^k \times [0, 1]$ is an ε_i -homotopy from $p_0|$ to $p_1g^{i-1}|$, by 3.5 there is a fibre preserving map over $\Delta^k \times [0, 1]$, $f^i: M \times \Delta^k \times [0, 1] \rightarrow B \times \Delta^k \times [0, 1]$ such that $f^i|M \times \Delta^k \times \{0\} = p_0$, $f^i|M \times \{1\} = p_1h^{i-1}$, $f^i|\partial M \times \Delta^k \times [0, 1] = pg^{i-1}|\partial M \times \Delta^k \times [0, 1]$, f^i is an ε_i -homotopy, and $f_t^i M \rightarrow B$ is an approximate fibration for each t in $\Delta^k \times [0, 1]$.

Now use 3.2 on f^i to obtain a homeomorphism $k^i: M \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1]$ such that k^i is fibre preserving over $\Delta^k \times [0, 1]$, $k^i|M \times \Delta^k \times \{0\} = id$, $f^i k^i$ is δ_i -close to $(f^i|M \times \Delta^k \times \{0\}) \times id_{[0,1]} = p_0 \times id_{[0,1]}$, and $f^i = g^i$ on a neighborhood of $\partial M \times \Delta^k \times [0, 1]$.

Finally, define $h^i: M \times \Delta^k \times [i, i+1] \rightarrow M \times \Delta^k \times [i, i+1]$ by setting $h_t^i = h_i^{i-1} k_{t-i}^i$ for each t in $\Delta^k \times [i, i+1]$. \square

Corollary 3.7. *Let M, B, p, p_0 , and p_1 be as above. Suppose further that there is a homeomorphism $G: \partial M \times \Delta^k \times [0, 1] \times [0, 1] \rightarrow \partial B \times \Delta^k \times [0, 1] \times [0, 1]$ such that*

- (1) $G = id$ on $\partial M \times \Delta^k \times [0, 1] \times \{0\}$ and $\partial M \times \Delta^k \times \{0\} \times [0, 1]$.
- (2) G is fibre preserving on $\Delta^k \times [0, 1] \times [0, 1]$, and
- (3) G is a controlled homeomorphism from $p|\partial M \times \Delta^k \times [0, 1]$ to $p_0 \times id_{[0,1]}|\partial M \times \Delta^k \times [0, 1]$.

Then there is a homeomorphism $H: M \times \Delta^k \times [0, 1] \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1] \times [0, 1]$ such that

- (1) $H = id$ on $M \times \Delta^k \times [0, 1] \times \{0\}$ and $M \times \Delta^k \times \{0\} \times [0, 1]$,
- (2) H is fibre preserving on $\Delta^k \times [0, 1] \times [0, 1]$,
- (3) $H|\partial M \times \Delta^k \times [0, 1] \times [0, 1] = G$, and
- (4) H is a controlled homeomorphism from p to $p_0 \times id_{[0,1]}$.

Proof. We obtain a map $q: M \times \Delta^k \times [0, 1]^2 \rightarrow B \times \Delta^k \times [0, 1]^2$ by rotating the map p through an angle of 90° , as in the proof of 2.5. From this construction we see that q is fibre preserving over $\Delta^k \times [0, 1]^2$, with each slice $q_t: M \rightarrow B$, t in $\Delta^k \times [0, 1]^2$, an approximate fibration such that

- (1) $q|M \times \Delta^k \times [0, 1] \times \{0\} = p$,
- (2) $q|M \times \Delta^k \times [0, 1] \times \{1\} = p_0 \times id_{[0,1]}$, and
- (3) $q|M \times \Delta^k \times \{0\} \times [0, 1] = p_0 \times id_{[0,1]}$.

Now we apply 3.6 to the map q . More precisely, let $\alpha: M \times \Delta^k \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1]$ be defined as $\alpha(t_1, t_2, t_3) = (t_1, t_3, t_2)$, and let $\beta: M \times \Delta^k \times [0, 1] \times [0, 1] \rightarrow M \times \Delta^k \times [0, 1] \times [0, \infty)$ be defined as $\beta(t_1, t_2, t_3, t_4) = (t_1, t_2, t_3, \frac{t_4}{1-t_4})$. Let $P^* = \alpha q \alpha^{-1}: M \times [0, 1] \times \Delta^k \rightarrow B \times [0, 1] \times \Delta^k$ and let $G^* = \beta G \beta^{-1}: \partial M \times [0, 1] \times \Delta^k \times [0, \infty) \rightarrow \partial B \times [0, 1] \times \Delta^k \times [0, \infty)$. It should be noted at this point that the previous theorem would require that G^* be defined on a neighborhood of $\partial(M \times [0, 1]) \times \Delta^k \times [0, \infty)$. This can be accomplished easily enough. First define G^* on $\partial(M \times [0, 1]) \times \Delta^k \times [0, \infty)$ by setting G^* equal to the identity on $M \times \{0\} \times \Delta^k \times [0, \infty)$, and if we collar ∂M in $M \times \{1\}$, we can define G^* to be $\beta G \beta^{-1}$ on this collar neighborhood and the identity on the complement of this neighborhood of $M \times \{1\}$. Now extend to a collar neighborhood of $\partial(M \times [0, 1]) \times \Delta^k \times [0, 1]$.

Applying 3.6, we get a homeomorphism $H^*: M \times [0, 1] \times \Delta^k \times [0, \infty) \rightarrow M \times [0, 1] \times \Delta^k \times [0, \infty)$ such that

- (1) $H^*|M \times [0, 1] \times \Delta^k \times \{0\} = id$,
- (2) H^* is fibre preserving over $\Delta^k \times [0, \infty)$ (actually over $[0, 1] \times \Delta^k \times [0, \infty)$),
- (3) $H^*|\partial(M \times [0, 1]) \times \Delta^k \times [0, \infty) = G^*$, and

(4) $(P_1^* \times id_{[0,\infty)})H^*: M \times [0,1] \times \Delta^k \times [0,\infty) \rightarrow B \times [0,1] \times \Delta^k \times [0,\infty)$ extends to a map $M \times [0,1] \times \Delta^k \times [0,\infty) \rightarrow B \times [0,1] \times \Delta^k \times [0,\infty)$ via P_0^* (i.e. H^* is a controlled homeomorphism from P_0^* to P_1^*).

Finally, let $H = \beta^{-1}H^*\beta: M \times \Delta^k \times [0,1] \times [0,1) \rightarrow M \times \Delta^k \times [0,1] \times [0,1)$, then

- (1) $H|M \times \Delta^k \times [0,1] \times \{0\} = id$,
- (2) H is fibre preserving over $\Delta^k \times [0,1] \times [0,1)$,
- (3) $H|\partial M \times \Delta^k \times [0,1] \times [0,1) = G$, and
- (4) H is a controlled homeomorphism from p to $p_0 \times id_{[0,1]}$.

In addition, since $H^* \models G^*$, $H|M \times \Delta^k \times \{0\} \times [0,1) = id$. \square

We are now ready to prove the main theorem of this section.

Proof. Theorem 3.1. The proof is a straightforward application of 3.7. The isotopy g_s is the s -level of a homeomorphism $g: \partial M \times \Delta^k \times [0,1] \times [0,1) \rightarrow \partial M \times \Delta^k \times [0,1] \times [0,1)$ which is the identity on $(\partial M \times \Delta^k \times [0,1] \times \{0\}) \cup (\partial M \times \Delta^k \times \{0\} \times [0,1))$, and which is fibre preserving over $\Delta^k \times [0,1] \times [0,1)$. This function g is a controlled homeomorphism from $h(p \times id_{[0,1]}|\partial M \times \Delta^k \times [0,1])$ to $p \times id_{[0,1]}$. Hence we get a continuous family of homeomorphisms $H_s: M \times \Delta^k \times [0,1] \rightarrow M \times \Delta^k \times [0,1]$, $0 \leq s < 1$, such that

- (1) $H_0 = id$,
- (2) H_s is an isotopy for each s ,
- (3) H_s is fibre preserving over Δ^k for each s ,
- (4) $H_s|\partial M \times \Delta^k \times [0,1] = G_s$ for each s , and
- (5) $(p \times id_{[0,1]})H$ converges uniformly to $h(p \times id_{[0,1]})$ as $s \rightarrow 1^-$. \square

Here is the corollary that is used in the previous section.

Corollary 3.8. *Let $p: M \times \Delta^k \times \mathbf{R} \rightarrow B \times \Delta^k \times \mathbf{R}$ be a k -parameter family of manifold approximate fibrations $M \times \mathbf{R} \rightarrow B \times \mathbf{R}$, $\dim(M) \geq 5$ such that $p|: \partial M \times \Delta^k \times \mathbf{R} \rightarrow \partial B \times \Delta^k \times \mathbf{R}$ is a k -parameter family of manifold approximate fibrations $\partial M \times \mathbf{R} \rightarrow \partial B \times \mathbf{R}$. Suppose that there is an isotopy $h: B \times \Delta^k \times \mathbf{R} \times I \rightarrow B \times \Delta^k \times \mathbf{R} \times I$ which is fibre preserving over Δ^k , and is supported over $[a,b] \subset \mathbf{R}$ (meaning that $h = id$ outside of the set $B \times \Delta^k \times [a,b] \times I$). For any interval (c,d) with $p^{-1}(B \times \Delta^k \times [a,b]) \subset M \times \Delta^k \times (c,d)$ and continuous family of isotopies $g_s: \partial M \times \Delta^k \times \mathbf{R} \times I \rightarrow \partial M \times \Delta^k \times \mathbf{R} \times I$, $0 \leq s < 1$, such that*

- (1) $g_0 = id$,
- (2) g_s is fibre preserving over Δ^k for all s ,
- (3) g_s is supported over $[c,d]$ for all s , and
- (4) $(p \times id_I)g_s$ converges uniformly to $h((p|\partial M \times \Delta^k \times \mathbf{R}) \times id_I)$ as $s \rightarrow 1^-$, there exists a continuous family of isotopies $H_s: M \times \Delta^k \times \mathbf{R} \times I \rightarrow M \times \Delta^k \times \mathbf{R} \times I$, $0 \leq s < 1$, such that

- (1) $H_0 = id$,
- (2) H_s is fibre preserving over Δ^k for all s ,
- (3) $H_s|\partial M \times \Delta^k \times \mathbf{R} \times I = g_s$ for all s ,
- (4) H_s is supported over $[c,d]$, and
- (5) $(p \times id_I)H_s$ converges uniformly to $h(p \times id_I)$ as $s \rightarrow 1^-$.

Proof. The restriction $p|: M \times \Delta^k \times [c,d] \rightarrow B \times \Delta^k \times \mathbf{R}$ is a manifold approximate fibration where $M \times \Delta^k \times [c,d]$ is a manifold with boundary $\partial(M \times \Delta^k \times [c,d]) = (\partial M \times \Delta^k \times [c,d]) \cup (M \times \Delta^k \times \partial[c,d])$. We define a continuous family of isotopies

on this boundary by $\bar{g}_s = g_s \cup id_{M \times \Delta^k \times \partial[c,d] \times I}$ which satisfies the hypotheses of 3.1. Consequently, we get a continuous family of isotopies $\bar{H}_s: M \times \Delta^k \times I \rightarrow M \times \Delta^k \times I$, $0 \leq s < 1$, such that

- (1) $\bar{H}_0 = id$,
- (2) \bar{H}_s is fibre preserving over Δ^k for each s ,
- (3) $\bar{H}_s|_{\partial(M \times \Delta^k \times [c,d])} = \bar{g}_s$, and
- (4) $(p| \times id_{[0,1]})\bar{H}_s$ converges uniformly to $h(p| \times id_{[0,1]})$ as $s \rightarrow 1^-$.

Now define H_s by $H_s = \bar{H}_s$ on $M \times \Delta^k \times [c,d] \times I$ and $H_s = id$ on the rest of $M \times \Delta^k \times \mathbf{R} \times I$. \square

REFERENCES

1. T. A. Chapman, *Approximation results in topological manifolds*, Mem. Amer. Math. Soc. **34** (1981). MR **83i**:57005
2. D. Coram and P. Duvall, *Approximate fibrations*, Rocky Mountain J. Math. **7** (1977), 275–288. MR **56**:1296
3. R. D. Edwards, *TOP regular neighborhoods*, handwritten manuscript, 1973.
4. S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, 1965. MR **31**:6202
5. C. B. Hughes, *Approximate fibrations on topological manifolds*, Michigan Math. J. **32** (1985), 167–183. MR **87e**:57025
6. ———, *DeLooping controlled pseudo-isotopies of Hilbert cube manifolds*, Topology Appl. **26** (1987), 175–191. MR **89a**:57023
7. ———, *Blocked approximate fibrations*, handwritten manuscript, 1987.
8. C. B. Hughes, L. R. Taylor, and E. B. Williams, *Bundle theories for topological manifolds*, Trans. Amer. Math. Soc. **319** (1990), 1–65. MR **91e**:57035
9. ———, *Manifold approximate fibrations are approximately bundles*, Forum Math. **3** (1991). MR **92k**:57040
10. ———, *Bounded homeomorphisms over Hadamard manifolds*, preprint, 1992.
11. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Math. Studies, 11, Van Nostrand, 1967. MR **36**:5942
12. E. Pedersen, *Regular neighborhoods in topological manifolds*, Michigan Math. J. **24** (1977), 177–183. MR **58**:2828
13. F. Quinn, *Ends of maps I*, Ann. of Math. **110** (1979) 275–331. MR **82k**:57009

DEPARTMENT OF MATHEMATICS, PRESBYTERIAN COLLEGE, CLINTON, SOUTH CAROLINA 29325
E-mail address: jjones@cs1.presby.edu